

Class 17 - Exponential Functions

Recall: Polynomial functions (linear & quadratic)

Rational functions

Review:

Properties of Exponents:

- Product rule: $a^m \cdot a^n = a^{m+n}$. Ex: $3^2 \cdot 3^4 = 3^6$
- Quotient rule: $\frac{a^m}{a^n} = a^m \div a^n = a^{m-n}$. Ex: $\frac{3^6}{3^2} = 3^6 \div 3^2 = 3^4$
- Power rule: $(a^m)^n = a^{m \cdot n}$. Ex: $(3^2)^4 = 3^8$
- Power of a product: $(a \cdot b)^n = a^n \cdot b^n$. Ex: $(4 \cdot 3)^2 = 4^2 \cdot 3^2$
- Power of a quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. Ex: $\left(\frac{4}{5}\right)^3 = \frac{4^3}{5^3}$
- Zero exponent rule: if $b \neq 0$, $b^0 = 1$. Ex: $2025^0 = 1$
- Negative exponent rule: $a^{-n} = \frac{1}{a^n}$. Ex: $2^{-3} = \frac{1}{2^3}$
- Fractional exponent rule: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$. Ex: $\sqrt{4} = \sqrt[2]{4^1} = 4^{\frac{1}{2}}$

Goal: use the lowest possible base

$$\begin{array}{l} \bullet 64 = 8^2 = 4^3 = 2^6 \\ \bullet \frac{1}{5^9} = 5^{-9} \end{array} \left\{ \begin{array}{l} \bullet \left(\frac{1}{3}\right)^{x-5} = 3^{-(x-5)} = 3^{5-x} \\ \bullet \frac{25}{\sqrt[4]{5^{x+1}}} = \frac{5^2}{5^{\frac{x}{4}}} = 5^{2-\frac{x}{4}} \end{array} \right\} \begin{array}{l} \bullet \sqrt{7} e^{\pi} \approx 61.2245 \\ \sqrt{(7)} e^{\wedge}(\pi) \end{array}$$

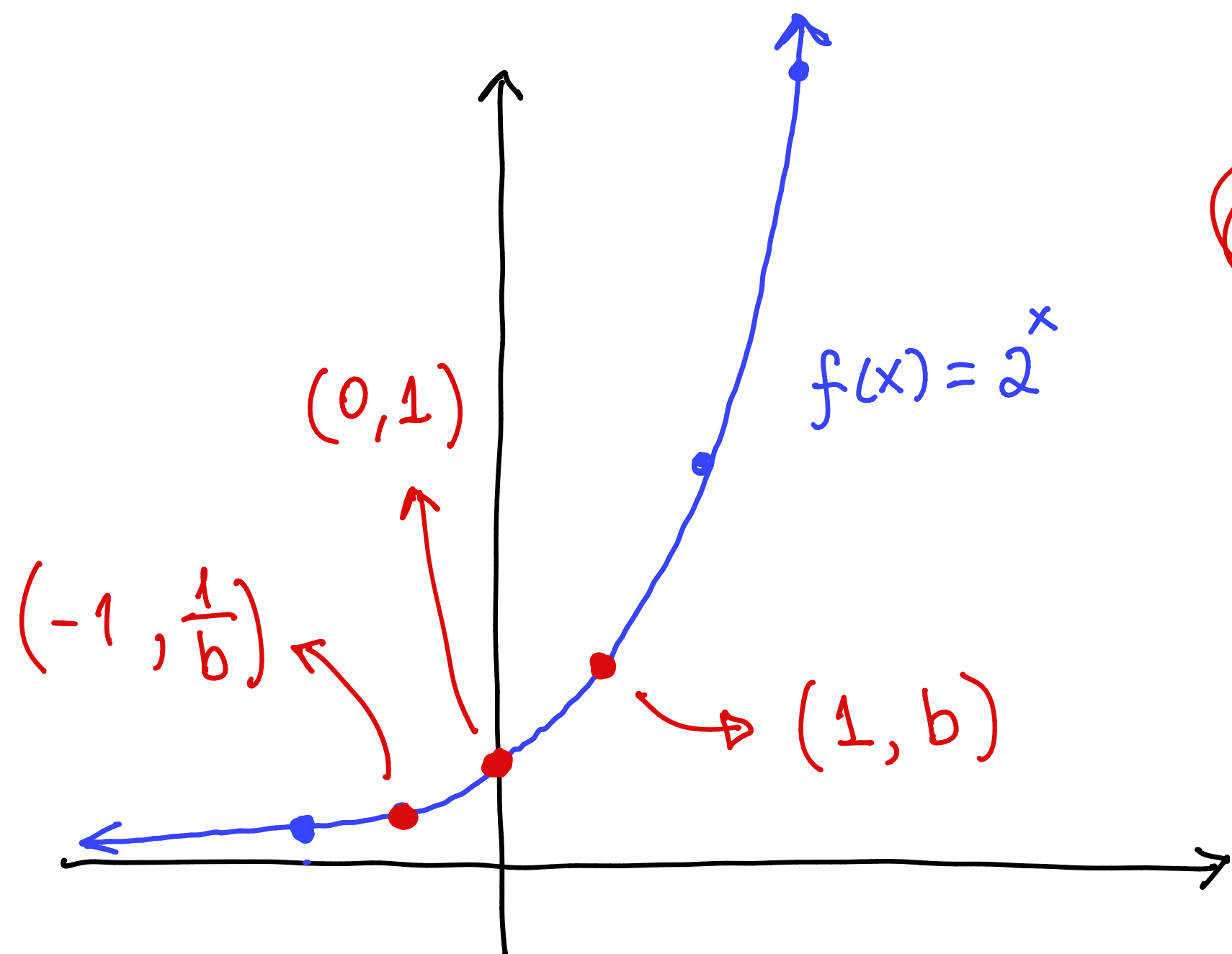
The number e:

	# of operations	Total
\$ → \$ + \$		
1 $\xrightarrow[+100\%]{1 \text{ year}}$ 2	1	2
1 $\xrightarrow[+50\%]{6 \text{ m}}$ 1.5 $\xrightarrow[+50\%]{6 \text{ m}}$ 2.25	2	2.25
1 $\xrightarrow[+25\%]{3 \text{ m}}$ 1.25 $\xrightarrow[+25\%]{3 \text{ m}}$ 1.56 $\xrightarrow[+25\%]{3 \text{ m}}$ 1.95 $\xrightarrow[+25\%]{3 \text{ m}}$ 2.44	4	2.44
1 $\xrightarrow[+8.3\% \text{ every mo.}]{\text{once every mo.}}$ 2.61	12	2.61
1 $\xrightarrow{+0.27\% \text{ every day}}$ 2.71	365	2.71

Continuous rate of interest: $e \approx 2.71828...$

Exponential function

$f(x) = b^x$, $b > 0$ & $b \neq 1$. Ex: $f(x) = 2^x$, $g(x) = 10^x$, $h(x) = e^x$.

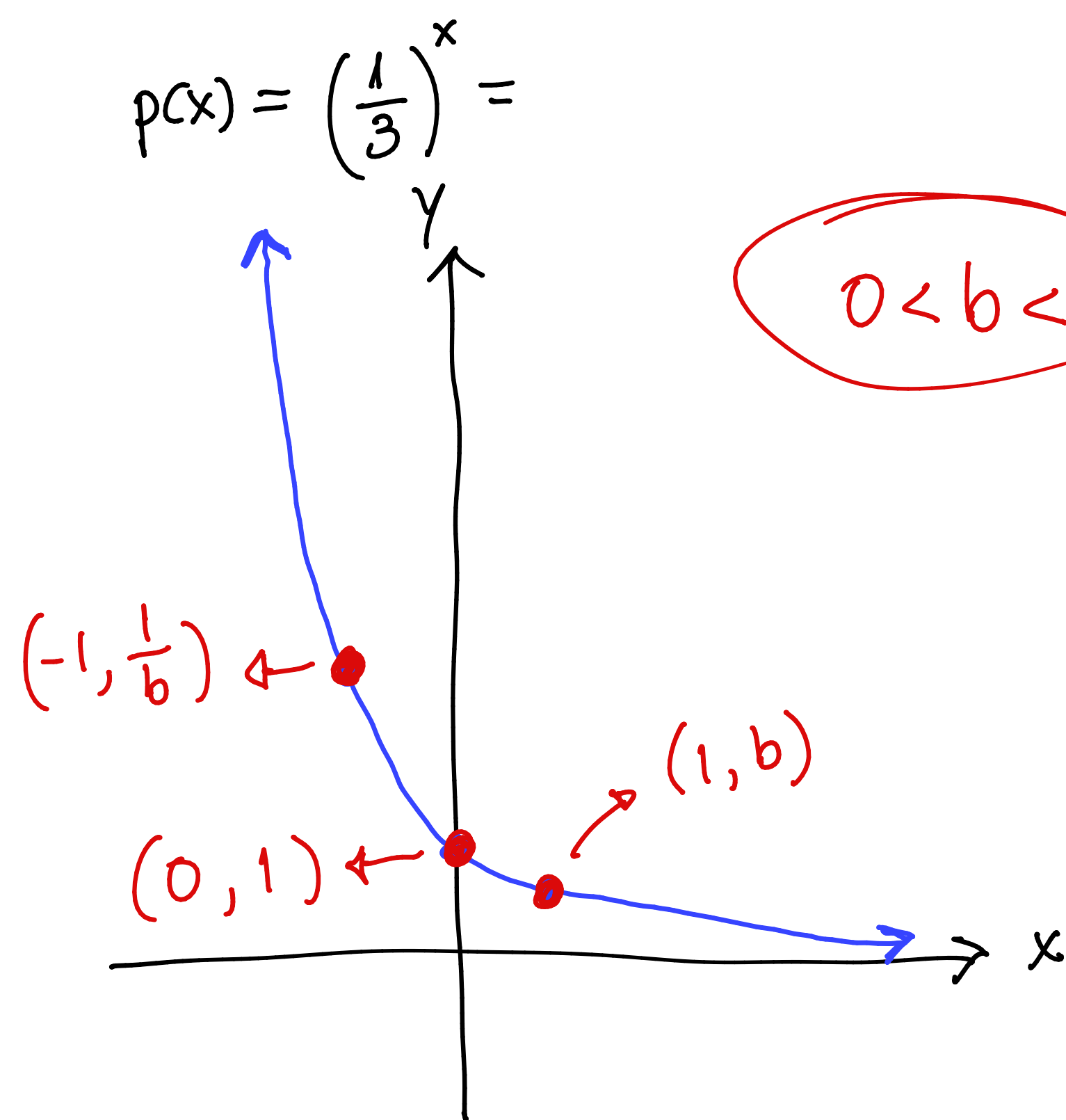


$b > 1$

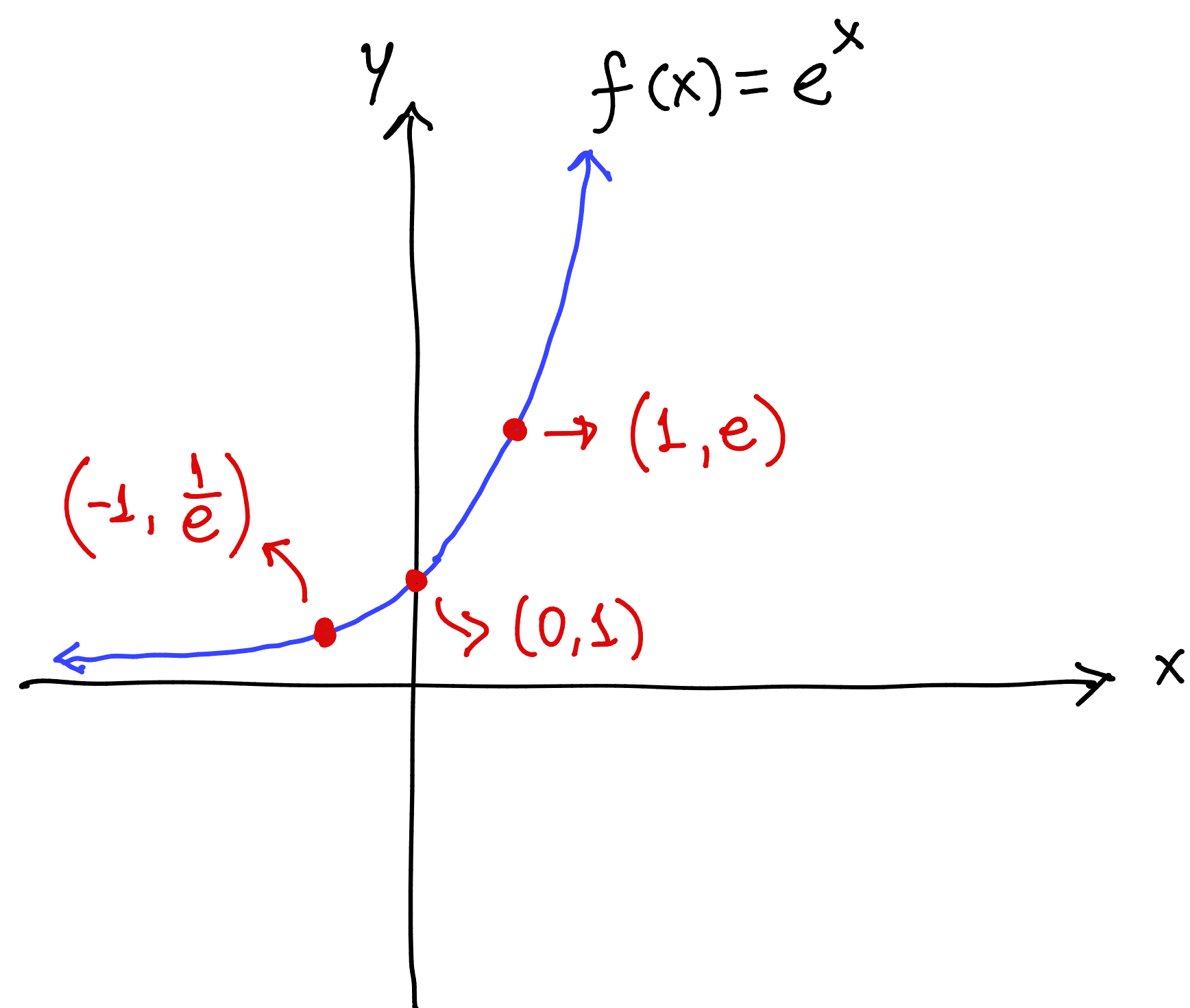
1-to-1

x	f(x)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4} = 0.25$
-1	$2^{-1} = \frac{1}{2^1} = 0.5$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

$$f(-2025) = \frac{1}{2^{2025}} = 0.00000001...$$



$$0 < b < 1$$



Solving exponential equations

$$b^u = b^v \Leftrightarrow u = v$$

Ex 0: $2^x = 2^3 \Leftrightarrow x = 3$ or $x^4 = 5^4 \Leftrightarrow x = 5$.

Ex 1: $e^x = \frac{1}{e^{11}} \Leftrightarrow e^x = e^{-11} \Leftrightarrow x = -11$. $\rightarrow$ Immediate solution

Ex 2: $\frac{1}{\sqrt[3]{25}} = 5^x \Leftrightarrow \frac{1}{5^{2/3}} = 5^x \Leftrightarrow 5^{-2/3} = 5^x \Leftrightarrow -\frac{2}{3} = x$.

Notice: $\sqrt[3]{25} = \sqrt[3]{5^2} = 5^{2/3}$

$\rightarrow$ linear equation

Ex 3: $27^x = \left(\frac{1}{3}\right)^{x-2} \Leftrightarrow (3^3)^x = 3^{-(x-2)} \Leftrightarrow 3^{3x} = 3^{-x+2}$

$\Leftrightarrow 3x = -x + 2 \Leftrightarrow \frac{4x}{4} = \frac{2}{4} \Leftrightarrow x = \frac{1}{2}$

$+x \quad +x$

Ex 4: $\frac{2^{x^2}}{8^x} = 16 \Leftrightarrow \frac{2^{x^2}}{(2^3)^x} = 2^4 \Leftrightarrow \frac{2^{x^2}}{2^{3x}} = 2^4 \Leftrightarrow 2^{x^2-3x} = 2^4$

$\Leftrightarrow x^2 - 3x = 4 \Leftrightarrow x^2 - 3x - 4 = 0$

$-4 \quad -4$

$\Leftrightarrow (x+1)(x-4) = 0 \Leftrightarrow x = -1$ or $x = 4$.

$$\text{Ex 5: } \frac{16}{\sqrt[4]{2^x}} = \frac{1}{8^x} \Leftrightarrow \frac{2^4}{2^{\frac{x}{4}}} = \frac{1}{(2^3)^x} \Leftrightarrow 2^{4-\frac{x}{4}} = 2^{-3x}$$

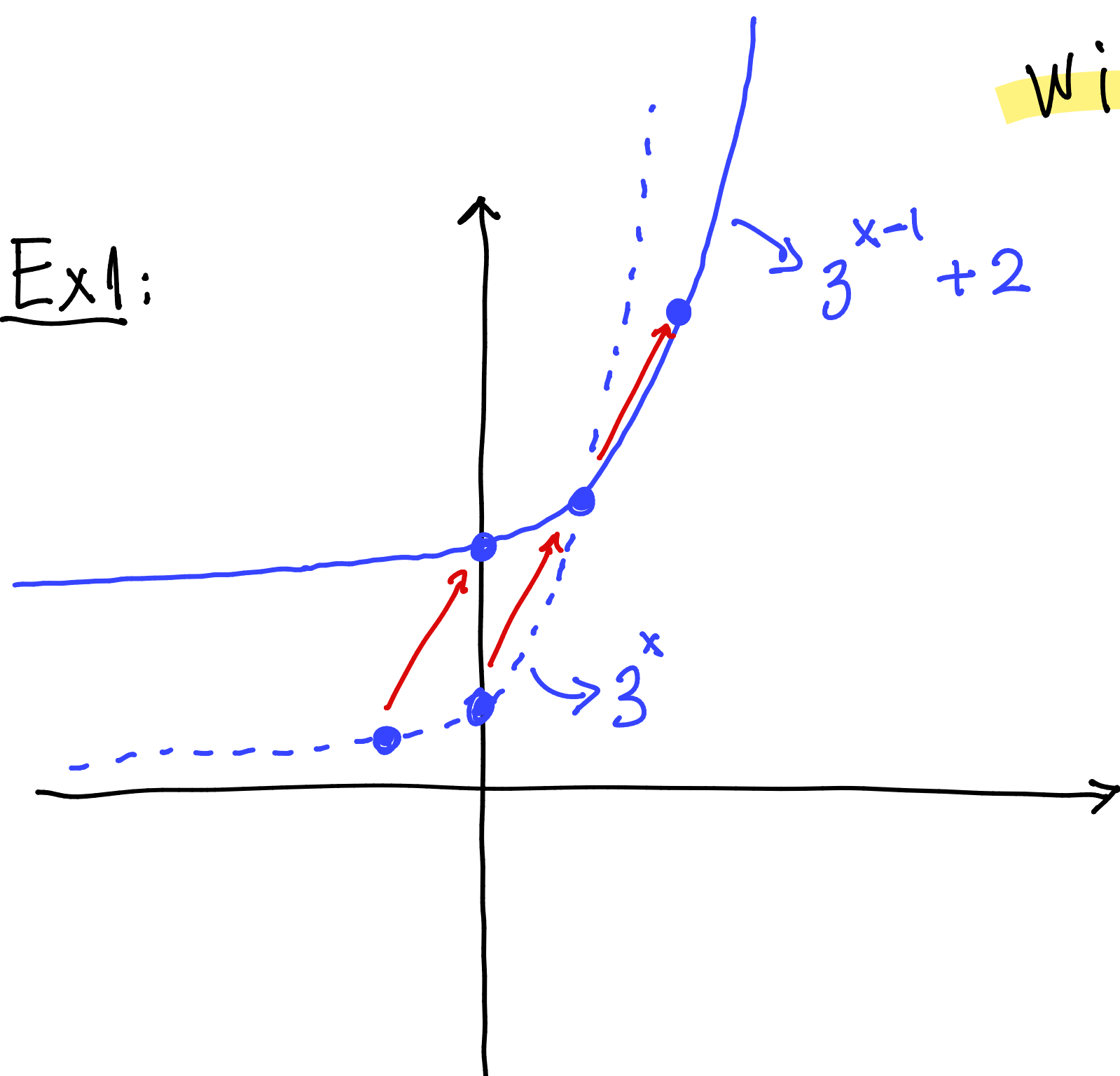
$$\Leftrightarrow \left(4 - \frac{x}{4}\right)^4 = (-3x)^4 \Leftrightarrow \underset{+x}{16} - \underset{+x}{x} = -12x$$

$$\Leftrightarrow \frac{16}{-11} = \frac{-11x}{-11} \Leftrightarrow \boxed{x = -\frac{16}{11}}$$

Graph of Exponential functions

with transformations

Ex1:



$$f(x) = 3^{x-1} + 2 \quad \begin{array}{l} \text{1 right} \\ \text{2 up} \end{array}$$

x	$3^x = y$
-1	$\frac{1}{3}$
0	1
1	3
<u>1</u>	<u>2</u>
1 →	2 ↑

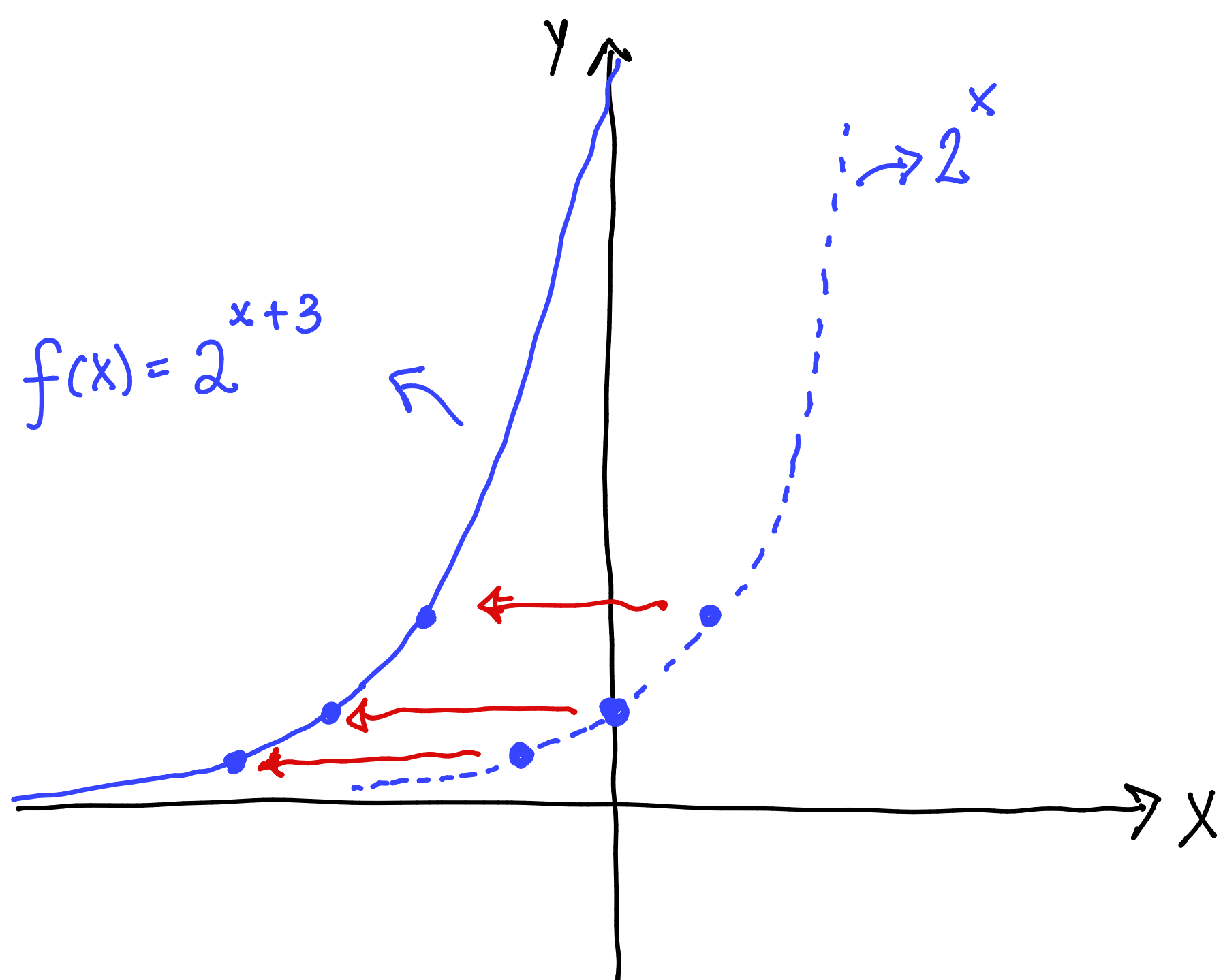
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x	$3^{x-1} + 2 = y$
0	$\frac{1}{3} + 2 = \frac{7}{3}$
1	1 + 2 = 3
2	3 + 2 = 5

$$\text{Ex2: } f(x) = 2^{x+3}$$



x	2^x
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
<u>3</u>	<u>4</u>
3 ←	NOTHING

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x	2^{x+3}
-4	$\frac{1}{2}$
-3	1
-2	2